

YOUNG MODULE MULTIPLICITIES AND CLASSIFYING THE INDECOMPOSABLE YOUNG PERMUTATION MODULES

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ABSTRACT. We study the multiplicities of Young modules as direct summands of permutation modules on cosets of Young subgroups. Such multiplicities have become known as the p -Kostka numbers. We classify the indecomposable Young permutation modules, and, applying the Brauer construction for p -permutation modules, we give some new reductions for p -Kostka numbers. In particular we prove that p -Kostka numbers are preserved under multiplying partitions by p , and strengthen a known reduction given by Henke, corresponding to adding multiples of a p -power to the first row of a partition.

The symmetric group permutation modules on the cosets of Young subgroups are known as the Young permutation modules. They play a central role in the representation theory of the symmetric group, and also in Schur algebras, relating representations of symmetric groups to polynomial representations of general linear groups. The indecomposable modules occurring in direct sum decompositions of Young permutation modules were parametrized by James, and have become known as the Young modules. In the semisimple case, the Young modules are the familiar Specht modules, and their multiplicities as direct summands of permutation modules (the Kostka numbers) have long been known by Young's rule. However, when the group algebra of the symmetric group is not semisimple, the multiplicities remain undetermined. It is well-known that the determination of these multiplicities is equivalent to determining the decomposition numbers of the symmetric groups and the Schur algebras.

Let \mathcal{S}_r be the symmetric group on r letters, and let k be a field of characteristic p . Both the Young permutation modules and the Young modules for \mathcal{S}_r are indexed by the partitions of r , and denoted by M^λ and Y^λ respectively. In this paper we focus on the p -Kostka numbers. That is, the multiplicities $[M^\lambda : Y^\mu]$ of a Young module Y^μ as a direct summand of M^λ for partitions λ, μ of r . Whilst a full determination of the p -Kostka numbers is out of reach, there are several known reduction formulae. The first appears to be Klyachko's multiplicity formula (see for example [16, Theorem 4.6.3(ii)]). Donkin [3, 3.6] has given an algorithmic description, related closely to Klyachko's formula, of when a Young module occurs as a direct summand of a Young permutation module. Other reductions have been given by Henke and Koenig in [10] and also by Fang, Henke, and Koenig in [5], using

Schur algebras and Ringel duality. In [5] it is shown that a certain complement construction on partitions preserves p -Kostka numbers, and as a consequence, row and column removal formulae are then deduced. Henke has determined the p -Kostka numbers corresponding to 2-part partitions in [9].

The paper is split into two halves. In the first half, we determine several new reduction formulae for p -Kostka numbers. The first of these shows that multiplication of partitions by p preserves the p -Kostka numbers as follows:

Theorem 1. *Let $\lambda, \mu \vdash r$. The following holds:*

$$[M^{p\lambda} : Y^{p\mu}] = [M^\lambda : Y^\mu].$$

This result, and further reductions are proved in section 2. The main method is to use the Brauer construction applied to p -permutation modules as developed by Broué, and the description of the Brauer quotients of Young modules given by Erdmann. With some analysis of the combinatorics of Young vertices we first prove Theorem 1, and then use this to determine a lower bound for certain p -Kostka numbers corresponding to adding p -power multiples of partitions (Theorem 13). Whilst this in general appears to be an inequality, we give certain conditions under which there is equality. One special case of this equality (Theorem 14) strengthens the reduction given by Henke in [9] which says the p -Kostka numbers are preserved under adding integer multiples of certain p -powers to the first parts of the partitions.

In the second half of the paper we take a different approach and classify the indecomposable Young permutation modules. We also determine precisely when a Young permutation module has direct summands occurring in more than one block. We summarise these results as:

Theorem 2. *Let $r \in \mathbb{N}$ and let λ be a partition of r , then*

- (a) *if p is odd, then M^λ is indecomposable if, and only if, one of the following holds:*
 - p divides r and λ is equal to (r) or $(r-1, 1)$;
 - p does not divide r and $\lambda = (r)$.
- (b) *if $p = 2$, then M^λ is indecomposable if, and only if, one of the following holds:*
 - r is odd and $\lambda = (r)$
 - r is even and $\lambda = (r)$ or λ is one of the n partitions $(r - k_i, k_i)$ where $2^n \leq r < 2^{n+1}$, and for each $1 \leq i \leq n$ the non-negative integer k_i is such that $2^{i-1} \leq k_i < 2^i$ and $k_i \equiv \frac{r-2^n}{2} \pmod{2}$.

Furthermore, if either p is odd, or $p = 2$ and r is odd, then M^λ is decomposable if, and only if, it has summands lying outside of the principal block. If $p = 2$ and r is

even then M^λ has direct summands lying outside of the principal block if, and only if $l(\lambda) \geq 3, r \geq 6$ and $\lambda \neq (r-2, 1, 1)$.

This has applications to the study of the endomorphism algebras of Young permutation modules. If λ is a partition of r with at most n parts then the Schur algebra $S(n, r)$ has an idempotent e_λ (described explicitly in [8]) for which $e_\lambda S(n, r)e_\lambda$ is isomorphic to the endomorphism algebra $\text{End}_{kS_r}(M^\lambda)$. Thus, studying the endomorphism algebras of Young permutation modules gives information on the structure of Schur algebras. By determining the indecomposable Young permutation modules, we have determined for which λ the algebra $e_\lambda S(n, r)e_\lambda$ is local.

Example 3. When the characteristic p is at least 3, it follows by Theorem 2 that the partitions of r indexing the indecomposable Young permutation modules are very easily understood. To demonstrate the ease of computation when $p = 2$, we give the following two examples.

- (1) We apply Theorem 2 to calculate the indecomposable Young permutation modules of degree 126 over a field of characteristic 2. In this case $n = 6$ and $r - 2^n = 126 - 2^n = 62$. Hence there are precisely seven partitions $(r - k, k)$ of 126 which correspond to indecomposable Young permutation modules, the first of these is (126) . The remaining six such partitions correspond to $(r - k_i, k_i)$ where $2^{i-1} \leq k_i < 2^i$ and $31 \equiv k_i \pmod{2^{i-1}}$ for $1 \leq i \leq n$. It follows that $k_1 = 1, k_2 = 3, k_3 = 7, k_4 = 15, k_5 = 31$, and $k_6 = 63$. Thus, the indecomposable Young permutation modules for kS_{126} are:

$$M^{(126)}, M^{(125,1)}, M^{(123,3)}, M^{(119,7)}, M^{(111,15)}, M^{(95,31)}, M^{(63,63)}.$$

- (2) Let $r = 2^n$ for some $n \in \mathbb{N}$ and let $\lambda \vdash r$. Then M^λ is indecomposable if, and only if, $\lambda = (r)$ or $\lambda = (r - 2^i, 2^i)$ for $0 \leq i \leq n - 1$.

The proof of Theorem 2 is the subject of section 3. This material can be read independently of section 2, except for relying on a particular calculation - namely Example 9. The partitions for which M^λ has summands in more than one block are found by analysis of the ordinary characters of the permutation modules using the Littlewood–Richardson rule. This, along with a dimension argument is enough to classify the indecomposable Young permutation modules over fields of odd characteristic. The remaining cases in characteristic $p = 2$ are solved by analysis of the 2-Kostka numbers for 2-part partitions as determined by Henke (see [9]). To conclude section 3 we show that the partitions indexing the indecomposable Young permutation modules in characteristic 2 are preserved by multiplication by 2 and also when adding certain 2-powers.

1. BACKGROUND

We now give a brief introduction to the background theory. The reader is referred to [11] and [13] for background on representations of symmetric groups, and [14] or [17] for representations of finite groups.

Fix $r \in \mathbb{N}$. If λ is a partition of r , then we write $\lambda \vdash r$, and $r = \sum_{i=1}^{\infty} \lambda_i = |\lambda|$ is the degree of λ . A composition of r differs from a partition in that the entries need not be nonincreasing. If λ is a composition of r , which is denoted by $\lambda \models r$, then the length of λ , denoted by $l(\lambda)$ is the number of non-zero parts of λ . Following the usual convention, in both partitions and compositions we allow zero entries at the end. Thus, $(r-1, 1, 0)$ is a 3-part partition of r , of length 2.

If $\lambda \vdash r$ and $\mu \vdash n$, then $\lambda + \mu$ is the partition of $r+n$ with i th part given by $\lambda_i + \mu_i$. Additionally, if $a \in \mathbb{N}$ and $\lambda \vdash r$, then $a\lambda$ is the partition of ar obtained by multiplying every part of λ by a . We say that a composition $\lambda \models r$ is divisible by $a \in \mathbb{N}$ if $\lambda = a\mu$ for some $\mu \models \frac{r}{a}$. The concatenation of two compositions λ and μ is denoted by $\lambda \bullet \mu$. A composition γ is a refinement of λ if there exist compositions $\gamma^{(i)} \models \lambda_i$ for each i , such that $\gamma = \gamma^{(1)} \bullet \dots \bullet \gamma^{(t)}$ where $t = l(\lambda)$. Furthermore, if p is prime, then a partition in which every non-zero part is a power of p is called a p -partition.

If $\lambda = (\lambda_1, \dots, \lambda_t) \models r$ with $\lambda_t \neq 0$, then a Young subgroup corresponding to λ , denoted \mathcal{S}_λ is any subgroup of \mathcal{S}_r which is conjugate to

$$\mathcal{S}_{\{1, \dots, \lambda_1\}} \times \mathcal{S}_{\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}} \times \dots \times \mathcal{S}_{\{r-\lambda_t+1, \dots, r\}}.$$

We assume a basic familiarity with the notion of λ -tabloids and λ -tableaux as defined in [11, §3] and follow the same notation.

Let F be a field. Then the Young permutation module M_F^λ is the permutation $F\mathcal{S}_r$ -module with F -basis the set of all λ -tabloids. The point stabiliser of any λ -tabloid is a Young subgroup corresponding to λ .

An important family of modules is that of the Specht modules. If t is a λ -tableau, with column stabiliser C_t , and $\{t\}$ is the λ -tabloid containing t , then define the polytabloid $e_t := \{t\}K_t \in M_F^\lambda$ where K_t is the signed column sum $\sum_{\sigma \in C_t} \text{sgn}(\sigma)\sigma$. The module generated by e_t is the Specht module S_F^λ . If F is a field of characteristic zero, then $\{S_F^\lambda \mid \lambda \vdash r\}$ is a complete set of mutually inequivalent irreducible modules for $F\mathcal{S}_r$.

Throughout this paper, all results concerning ordinary representation theory (over fields of characteristic zero) will be stated in terms of characters. If F is a field of characteristic zero, then we denote the character of M_F^λ by ξ^λ , the character of S_F^λ by χ^λ and the usual inner product on characters by $\langle \cdot, \cdot \rangle$.

The multiplicities $\langle \xi^\lambda, \chi^\mu \rangle$ are well known by Young's rule as follows:

Theorem 4 (Young's Rule [11, 14.1]). *The multiplicity $\langle \xi^\lambda, \chi^\mu \rangle$ where $\lambda, \mu \vdash r$ is equal to the number of semistandard μ -tableaux of type λ .*

From now on, all modules are defined over a field k of characteristic $p > 0$.

The indecomposable direct summands of M^λ were parameterized by James. Specifically, in any direct sum decomposition of M^λ , there is precisely one direct summand containing S^λ as a submodule. Such a summand is defined up to isomorphism, and is known as the Young module corresponding to λ , denoted by Y^λ . Moreover, every indecomposable direct summand of M^λ is isomorphic to a Young module.

Theorem 5 ([12]). *Let $\lambda \vdash r$, then there exists a family of self-dual indecomposable modules $\{Y^\mu \mid \mu \vdash r\}$ and non-negative integers $[M^\lambda : Y^\mu]$ for $\lambda, \mu \vdash r$ such that*

- (a) $Y^\lambda \cong Y^\mu$ if, and only if, $\lambda = \mu$;
- (b) $M^\lambda \cong \bigoplus_{\mu \vdash r} [M^\lambda : Y^\mu] Y^\mu$;
- (c) $[M^\lambda : Y^\lambda] = 1$, and if $[M^\lambda : Y^\mu] \neq 0$ then $\mu \trianglerighteq \lambda$;

We note in particular that since $M^{(1^r)} \cong k\mathcal{S}_r$, it follows that every projective indecomposable $k\mathcal{S}_r$ -module is isomorphic to a Young module. A Young module Y^λ is projective if, and only if λ is p -restricted, that is, if, and only if, $\lambda_i - \lambda_{i+1} < p$ for every i .

The Nakayama conjecture (first proved by Brauer and Robinson in [1] and [18]) determines the blocks of $k\mathcal{S}_r$ in terms of p -cores of partitions. Two Specht modules S^λ and S^μ lie in the same block of $k\mathcal{S}_r$ if, and only if, λ and μ have the same p -core. Thus, to each block B of $k\mathcal{S}_r$ is associated a p -core $\tau \vdash r - pw$ where w is a non-negative integer called the weight of the block. A Specht module S^λ , irreducible character χ^λ , or a Young module Y^λ lie in B if, and only if, the corresponding partition λ has p -core τ .

2. SOME REDUCTIONS FOR p -KOSTKA NUMBERS

Let G be a finite group. Let M be an indecomposable kG -module, and H a subgroup of G , then M is said to be relatively H -projective if there exists a kH -module N such that M is a direct summand of the induced module $N \otimes_{kH} kG = N \uparrow^G$. J.A.Green ([7]) proved that every indecomposable kG -module M defines a subgroup Q of G such that if H is a subgroup of G , then M is relatively H -projective, if, and only if, H contains a conjugate of Q . Such a subgroup is always a p -group, is called a vertex of M and is defined uniquely up to conjugacy in G . If H is a vertex of M and N is an indecomposable kH -module such that M is a direct summand of $N \uparrow^G$, then N is called a source of M , and N is well-defined up to conjugacy in $N_G(H)$.

We make extensive use of the Brauer construction applied to p -permutation modules as developed by Broué in [2], which we describe below.

A kG -module M is said to be a p -permutation module if, for every p -subgroup P of G , the restriction of M to P is a permutation kP -module. If M is a p -permutation module then every direct summand of M is a p -permutation module. An indecomposable kG -module U is a p -permutation module if, and only if, U has trivial source, or equivalently, U is a direct summand of a permutation module.

The Brauer construction for kG -modules is defined as follows. Let M be a kG -module, and Q a subgroup of G . We denote by M^Q the set of elements of M which are fixed under the action of Q . If P is a subgroup of Q , then the relative trace map $\text{Tr}_P^Q : M^P \rightarrow M^Q$ is defined by

$$\text{Tr}_P^Q(m) = \sum_i mg_i,$$

where the sum runs over a set of right coset representatives for P in Q .

The Brauer quotient of M with respect to Q is defined to be the $N_G(Q)/Q$ -module

$$M(Q) := M^Q / \sum_{P < Q} \text{Tr}_P^Q(M^P).$$

The parameterisation of indecomposable p -permutation kG -modules is as follows:

Theorem 6 ([2, (3.2)], The Broué correspondence). *An indecomposable p -permutation module M has vertex P if, and only if, $M(P)$ is a non-trivial projective $kN_G(P)/P$ -module. Moreover,*

- (a) *The correspondence $M \rightarrow M(P)$ induces a bijection between the isomorphism classes of indecomposable p -permutation kG -modules with vertex P and the isomorphism classes of indecomposable projective $kN_G(P)/P$ -modules.*
- (b) *Let M and U be p -permutation kG -modules, and assume also that U is indecomposable with vertex P , then U is a direct summand of M if, and only if, $U(P)$ is a direct summand of $M(P)$, and they occur with the same multiplicities.*

Let M be a permutation kG -module, and P a p -subgroup of G . In this case, let B be a permutation basis for M , and let B^P be the set of elements of B which are fixed under the action of P . Then,

$$M^P \cong \langle B^P \rangle \oplus \sum_{Q < P} \text{Tr}_Q^P(M^Q) \text{ as a module for } N_G(P).$$

It follows that $M(P)$ is isomorphic to the linear span of B^P as a module for $N_G(P)/P$.

We return now to the case of Young modules for symmetric groups. Grabmeier proved in [6] that every Young module has an associated Young vertex. That is, if $r \in \mathbb{N}$, and $\lambda \vdash r$, then there exists a Young subgroup \mathcal{S}_ν such that the

Young module Y^λ is relatively \mathcal{S}_μ -projective if, and only if, \mathcal{S}_ν is \mathcal{S}_r -conjugate to a subgroup of \mathcal{S}_μ . It follows that for every Young module Y^λ there is a unique partition ν such that \mathcal{S}_ν is a Young vertex of Y^λ .

We recall now that a partition $\lambda = (\lambda_1, \dots, \lambda_t)$ is p -restricted if $\lambda_i - \lambda_{i+1} < p$ for every $i = 1, \dots, t-1$ and $\lambda_t < p$. With this notation, every partition λ has a unique p -adic expansion $\lambda = \sum_i \lambda(i)p^i$ where the $\lambda(i)$ are all p -restricted.

Let λ be a partition of r , then Grabmeier describes a Young vertex of Y^λ as follows: Let λ have p -adic expansion $\lambda = \sum_i \lambda(i)p^i$, and for each i , let $r_i = |\lambda(i)|$. Define ρ to be the partition of r with r_i parts equal to p^i for each i , then \mathcal{S}_ρ is a Young vertex of Y^λ . Grabmeier proved ([6, 4.7]) that if \mathcal{S}_ν is a Young vertex of Y^λ then a Sylow p -subgroup of \mathcal{S}_ν is a vertex of Y^λ .

If ν is a p -partition, and P_ν is a Sylow p -subgroup of \mathcal{S}_ν then P_ν and \mathcal{S}_ν have the same orbits for the natural action on $\{1, \dots, r\}$. In particular, P_ν is the trivial subgroup if, and only if, $\nu = (1^r)$. It follows that a Young module Y^λ is projective if, and only if, λ is p -restricted.

We will now describe the Brauer quotient of a Young module: We denote the outer tensor product by \boxtimes . That is, if G and H are finite groups and M is a kG -module, N a kH -module, then $M \boxtimes N$ is the $k(G \times H)$ -module with underlying vector space $M \otimes_k N$, and action given by linearly extending

$$(m \otimes n)(g, h) = mg \otimes nh.$$

Erdmann ([4]) parametrized the Young modules using only the representation theory of symmetric groups, and in the process described the Brauer quotients of Young modules as follows:

Theorem 7 ([4]). *Let ρ be a p -partition of r , and for each i , let r_i be the number of times p^i occurs as a part of ρ . Let P be a Sylow p -subgroup of \mathcal{S}_ρ .*

(a) $N_{\mathcal{S}_r}(\mathcal{S}_\rho) = N_{\mathcal{S}_r}(P)\mathcal{S}_\rho$ and if β is the composition (r_0, r_1, \dots) , then

$$\mathcal{S}_\beta \cong N_{\mathcal{S}_r}(\mathcal{S}_\rho)/\mathcal{S}_\rho \cong N_{\mathcal{S}_r}(P)/N_{\mathcal{S}_r}(P).$$

(b) Let $\lambda \vdash r$, then $N_{\mathcal{S}_r}(P)/P$ acts trivially on $Y^\lambda(P)$, so we may view $Y^\lambda(P)$ as a module for \mathcal{S}_β ;

(c) If \mathcal{S}_ρ is a Young vertex of Y^λ then P is a vertex of Y^λ , and

$$Y^\lambda(P) \cong Y^{\lambda(0)} \boxtimes \dots \boxtimes Y^{\lambda(t)}$$

where $\sum_{i=0}^t \lambda(i)p^i$ is the p -adic expansion of λ .

Throughout this section we make extensive use of the Broué correspondence to determine p -Kostka numbers. Recall that by Theorem 6, if $\lambda, \mu \vdash r$ and P is a

vertex of Y^μ , then

$$(1) \quad [M^\lambda : Y^\mu] = [M^\lambda(P) : Y^\mu(P)].$$

Thus, we must understand the Brauer quotient of the Young permutation modules with respect to Sylow p -subgroups of a Young subgroup \mathcal{S}_ν where ν is a p -partition. We follow the approach of Erdmann in [4].

Let $\mu \vdash r$, with p -adic expansion $\sum_{i=0}^t \mu(i)p^i$. Let ρ be the p -partition such that \mathcal{S}_ρ is a Young vertex of Y^μ , and P a Sylow p -subgroup of \mathcal{S}_ρ . A permutation basis for M^λ can be taken as the set of all λ -tabloids. That is, the set of row equivalence classes of λ -tableaux. We use the standard notation as described in [11, §3]. As such, we may identify $M^\lambda(P)$ with the linear span of the λ -tabloids which are fixed by P . A λ -tabloid is fixed by P if, and only if, each row is a union of P -orbits. Since the \mathcal{S}_ρ orbits on $\{1, \dots, r\}$ coincide with the P -orbits, it follows that each λ -tabloid which is fixed by P is also fixed by \mathcal{S}_ρ . Thus, \mathcal{S}_ρ acts trivially on $M^\lambda(P)$, and the structure of $M^\lambda(P)$ as a module for $N_{\mathcal{S}_r}(P)/P$ is the same as the structure of $M^\lambda(P)$ as a module for $N_{\mathcal{S}_r}(\mathcal{S}_\rho)/N_{\mathcal{S}_\rho}(P) \cong \mathcal{S}_\beta$ (where β is given as in Theorem 7). Since \mathcal{S}_β acts by permuting the \mathcal{S}_ρ -orbits of the same length amongst themselves, $M^\lambda(P)$ is a permutation module for \mathcal{S}_β . It follows from this construction that $M^\lambda(P)$ is a direct sum of outer tensor products of Young permutation modules

$$M^{\gamma^{(0)}} \boxtimes \cdots \boxtimes M^{\gamma^{(s)}}$$

for some refinement $\gamma = \gamma^{(0)} \bullet \cdots \bullet \gamma^{(s)}$ of β .

Assume $M^\lambda(P) \cong \bigoplus_{\gamma \models \beta} a_\gamma M^{\gamma^{(0)}} \boxtimes \cdots \boxtimes M^{\gamma^{(s)}}$, for some non-negative integers a_γ , as a module for \mathcal{S}_β . From $[M^\lambda : Y^\mu] = [M^\lambda(P) : Y^\mu(P)]$, and the form for $Y^\mu(P)$ in Theorem 7, follows:

$$(2) \quad [M^\lambda(P) : Y^\mu(P)] = \sum_{\gamma \models \beta} a_\gamma \prod_{i=0}^s [M^{\gamma^{(i)}} : Y^{\mu(i)}].$$

Thus, provided one knows the p -Kostka numbers corresponding to projective Young modules for all partitions of degree less than r , then the Brauer Morphism and the Broué correspondence is sufficient to determine the non-projective p -Kostka numbers in degree r .

Remark 8. In fact, Erdmann ([4, Proposition 1]) describes $M^\lambda(P)$ explicitly as the direct sum

$$M^\lambda(P) \cong \bigoplus_{\gamma} M^{\gamma^{(0)}} \boxtimes \cdots \boxtimes M^{\gamma^{(s)}},$$

where the sum is over all refinements $\gamma = \gamma^{(0)} \bullet \cdots \bullet \gamma^{(s)}$ of β where $\gamma^{(i)}$ is an improper partition of β_{i+1} for each i , and where $\sum_{i=0}^s \gamma^{(i)} p^i = \lambda$. Substituting this expression into (2), one obtains the Klyachko multiplicity formula. Whilst this

seems to imply that many of the results in this section can be proved using the Klyachko formula (for example this is certainly the case for Theorem 1) it seems to be easier to work with the Brauer construction in many cases.

The object of this is to provide a method for calculating p -Kostka numbers. An example will be illuminating. Recall that every integer r has a p -adic expansion, and we write this as $r = \sum_{i=0}^{\infty} [r]_i p^i$.

Example 9. We consider the case $p = 2$ and want to calculate a direct sum decomposition of $M^{(r-2,1,1)}$ for r even. We may assume $r \geq 4$. Recall that

$$M^{(r-2,1,1)} \cong Y^{(r-2,1,1)} \oplus \bigoplus_{\mu \triangleright (r-2,1,1)} [M^{(r-2,1,1)} : Y^\mu] Y^\mu.$$

The dominance lattice for partitions $\mu \trianglerighteq (r-2, 1, 1)$ is as follows:

$$(r) \triangleright (r-1, 1) \triangleright (r-2, 2) \triangleright (r-2, 1, 1).$$

It follows that to determine the decomposition of $M^{(r-2,1,1)}$ we need only concern ourselves with the multiplicities of the Young modules corresponding to the partitions $(r-2, 2)$, $(r-1, 1)$ and (r) . We first calculate $[M^{(r-2,1,1)} : Y^{(r-1,1)}]$. The 2-adic expansion of $(r-1, 1)$ is

$$(r-1, 1) = (1, 1) + \sum_{i=1}^t ([r-2]_i) 2^i.$$

It follows that a Young vertex of $Y^{(r-1,1)}$ is of the form \mathcal{S}_ρ where ρ is a p -partition of r , having exactly two parts equal to 1. Let P be a Sylow p -subgroup of \mathcal{S}_ρ , then P is a vertex of $Y^{(r-1,1)}$, and the set consisting of the tabloids

$$\begin{array}{c} \overline{1 \dots r-2} \\ \overline{r-1} \\ \hline r \end{array}, \quad \begin{array}{c} \overline{1 \dots r-2} \\ \overline{r} \\ \hline r-1 \end{array}$$

is a basis for $M^{(r-2,1,1)}(P)$ over k . The structure of $M^{(r-2,1,1)}(P)$ as a $N_{\mathcal{S}_r}(P)/P$ -module is the same as it has as a module for $N_{\mathcal{S}_r}(P)/N_{\mathcal{S}_\rho}(P)$. This group is isomorphic to $\mathcal{S}_{(2,1^{t-2})} \cong C_2$. Under this identification a generator of C_2 acts by permuting the entries $r-1$ and r and fixing entries $1, \dots, r-2$ in the tabloids above. We have demonstrated that $M^{(r-2,1,1)}(P) \cong k\mathcal{S}_{(2,1^{t-2})} \cong Y^{(1,1)} \boxtimes Y^{(1)} \boxtimes \dots \boxtimes Y^{(1)}$ as a module for $\mathcal{S}_{(2,1^{t-2})}$. It follows by the Broué correspondence that $M^{(r-2,1,1)}$ has exactly one indecomposable summand with vertex P , namely $Y^{(r-1,1)}$, and $[M^{(r-2,1,1)} : Y^{(r-1,1)}] = 1$.

Next, let \mathcal{S}_ν be the Young vertex of $Y^{(r-2,2)}$. Since all parts of $(r-2, 2)$ are divisible by 2, it follows that all parts of ν are divisible by 2. Hence there are no $(r-2, 1, 1)$ -tabloids which are fixed by the action of \mathcal{S}_ν . It follows that if Q is a Sylow 2-subgroup of \mathcal{S}_ν then $M^{(r-2,1,1)}(Q) = 0$, and hence $M^{(r-2,1,1)}$ has no

summands with vertex Q . In particular $[M^{(r-2,1,1)} : Y^{(r-2,2)}] = 0$. An identical line of reasoning proves that $[M^{(r-2,1,1)} : Y^{(r)}] = 0$. Hence we have determined the direct sum decomposition:

$$M^{(r-2,1,1)} \cong Y^{(r-2,1,1)} \oplus Y^{(r-1,1)}.$$

Some of the above argument generalises as follows:

Lemma 10. *Let r be a positive integer divisible by p . If λ is a partition of r not divisible by p , and μ is a partition of r divisible by p , then $[M^\lambda : Y^\mu] = 0$.*

Proof. Since p divides every part of μ , it follows that if ρ is a p -partition of r such that \mathcal{S}_ρ is a Young vertex of Y^μ , then p divides every part of ρ . Let P be a Sylow p -subgroup of \mathcal{S}_ρ , then P is a vertex of Y^μ . Since the orbits of P all have length divisible by p , it follows that no λ -tabloid is fixed by P . Thus, $M^\lambda(P) = 0$, and in particular, $[M^\lambda : Y^\mu] = 0$. \square

Next, we investigate the effect of multiplying partitions by p on the p -Kostka numbers.

Proposition 11. *Let $r \in \mathbb{N}$ and let $\mu, \nu \vdash r$ such that \mathcal{S}_ν is a Young vertex of Y^μ . Then $\mathcal{S}_{p\nu}$ is a Young vertex of $Y^{p\mu}$, and if P_ν and $P_{p\nu}$ are Sylow p -subgroups of \mathcal{S}_ν and $\mathcal{S}_{p\nu}$ respectively, then there exists a composition β such that*

$$N_{\mathcal{S}_r}(P_\nu)/N_{\mathcal{S}_\nu}(P_\nu) \cong \mathcal{S}_\beta \cong N_{\mathcal{S}_{pr}}(P_{p\nu})/N_{\mathcal{S}_{p\nu}}(P_{p\nu}).$$

Under this identification, the following hold:

- (a) $Y^\mu(P_\nu) \cong Y^{\mu(0)} \boxtimes \cdots \boxtimes Y^{\mu(s)} \cong Y^{p\mu}(P_{p\nu})$ as modules for \mathcal{S}_β .
- (b) Let $\lambda \vdash r$, then $M^\lambda(P_\nu) \cong M^{p\lambda}(P_{p\nu})$ as modules for \mathcal{S}_β .

Proof. If $\mu = \sum_{i=0}^s \mu(i)p^i$ is the p -adic expansion of μ , then the p -adic expansion of $p\mu$ is $\sum_{i=0}^s \mu(i)p^{i+1}$. Thus, if ν is the partition with $|\mu(i)|$ parts equal to p^i then \mathcal{S}_ν is a Young vertex of Y^μ and $\mathcal{S}_{p\nu}$ is a Young vertex of $Y^{p\mu}$.

Let P_ν and $P_{p\nu}$ be Sylow p -subgroups of \mathcal{S}_ν and $\mathcal{S}_{p\nu}$ respectively. If $\beta := (|\mu(0)|, |\mu(1)|, \dots, |\mu(s)|)$, then by Theorem 7(a),

$$N_{\mathcal{S}_r}(P_\nu)/N_{\mathcal{S}_\nu}(P_\nu) \cong N_{\mathcal{S}_r}(\mathcal{S}_\nu)/\mathcal{S}_\nu \cong \mathcal{S}_\beta$$

and similarly

$$N_{\mathcal{S}_{pr}}(\mathcal{S}_{p\nu})/\mathcal{S}_{p\nu} \cong N_{\mathcal{S}_{pr}}(P_{p\nu})/N_{\mathcal{S}_{p\nu}}(P_{p\nu}) \cong \mathcal{S}_\beta.$$

Statement (a) follows directly from the expression for the p -adic expansion of $p\mu$, and Theorem 7.

Recall that there is a basis \mathcal{B}_λ of $M^\lambda(P_\nu)$ consisting of the λ -tabloids with rows given by unions of \mathcal{S}_ν -orbits. Similarly there is a basis $\mathcal{B}_{p\lambda}$ of $M^{p\lambda}(P_{p\nu})$ consisting

of the $p\lambda$ -tabloids with rows given by unions of $\mathcal{S}_{p\nu}$ -orbits. Both $M^\lambda(P_\nu)$ and $M^{p\lambda}(P_{p\nu})$ are modules for $k\mathcal{S}_\beta$, with the corresponding action of \mathcal{S}_β on tabloids given by extending the action of \mathcal{S}_β on the \mathcal{S}_ν and $\mathcal{S}_{p\nu}$ orbits respectively. If $l(\nu) = s$ then we label the \mathcal{S}_ν orbits (respectively $\mathcal{S}_{p\nu}$ -orbits) by O_1, \dots, O_s (respectively $\bar{O}_1, \dots, \bar{O}_s$), so $O_i = \{\sum_{j=1}^i \nu_j + 1, \dots, \sum_{j=1}^{i+1} \nu_j\}$ (respectively $\bar{O}_i = \{\sum_{j=1}^i (p\nu_i) + 1, \dots, \sum_{j=1}^{i+1} (p\nu_j)\}$). Let x be a tabloid in \mathcal{B}_λ . If the i th row of x consists of $\bigcup_{j=1}^m O_{i,j}$ for each i , then there is a unique $p\lambda$ -tabloid $\bar{x} \in \mathcal{B}_{p\lambda}$ with i th row given by $\bigcup_{j=1}^m \bar{O}_{i,j}$ for each i . This defines a one-to-one correspondence between \mathcal{B}_λ and $\mathcal{B}_{p\lambda}$. Since \mathcal{S}_β permutes orbits of the same length amongst themselves, this correspondence induces an isomorphism $M^\lambda(P_\nu) \cong M^{p\lambda}(P_{p\nu})$ of \mathcal{S}_β -modules. \square

Proof of Theorem 1. We follow the setup of Proposition 11. Then

$$[M^{p\lambda} : Y^{p\mu}] = [M^{p\lambda}(P_{p\nu}) : Y^{p\mu}(P_{p\nu})] = [M^\lambda(P_\nu) : Y^\mu(P_\nu)] = [M^\lambda : Y^\mu],$$

where the middle equality follows from the isomorphisms (as $k\mathcal{S}_\beta$ -modules) in Proposition 11. \square

We compare now multiplicities of direct summands of $M^{\lambda+p^n\alpha}$ with multiplicities of direct summands of M^λ and M^α , for partitions λ and α .

Proposition 12. *Let $a, r \in \mathbb{N}$, and let $\mu \vdash r$, $\delta \vdash a$. Suppose μ has p -adic expansion $\sum_{i=0}^s \mu(i)p^i$ and let $n > s$. Let ρ, γ be partitions such that $\mathcal{S}_\rho, \mathcal{S}_\gamma$ are Young vertices of Y^μ, Y^δ respectively. Assume also that $N_{\mathcal{S}_r}(\mathcal{S}_\rho)/\mathcal{S}_\rho \cong \mathcal{S}_\beta$, and $N_{\mathcal{S}_r}(\mathcal{S}_\gamma)/\mathcal{S}_\gamma \cong \mathcal{S}_\eta$ where β and η are as in Theorem 7.*

(a) *If $\nu = \rho \bullet (p^n\gamma)$, then \mathcal{S}_ν is a Young vertex of $Y^{\mu+p^n\delta}$ and*

$$N_{\mathcal{S}_{r+a_p^n}}(\mathcal{S}_\nu)/\mathcal{S}_\nu \cong \mathcal{S}_{\beta \bullet \eta} \cong \mathcal{S}_\beta \times \mathcal{S}_\eta;$$

(b) *Let P_ρ and $P_{p^n\gamma}$ be Sylow p -subgroups of \mathcal{S}_ρ and $\mathcal{S}_{p^n\gamma}$ respectively, with disjoint supports. Then $P_\nu := P_\rho \times P_{p^n\gamma}$ is a Sylow p -subgroup of \mathcal{S}_ν , and*

$$Y^{\mu+p^n\delta}(P_\nu) \cong Y^\mu(P_\rho) \boxtimes Y^{p^n\delta}(P_{p^n\gamma}) \text{ as a module for } \mathcal{S}_\beta \times \mathcal{S}_\eta;$$

(c) *Let $\lambda \vdash r$ and $\alpha \vdash a$. There exists a direct summand N of $M^{\lambda+p^n\alpha}(P_\nu)$ such that $N \cong M^\lambda(P_\rho) \boxtimes M^{p^n\alpha}(P_{p^n\gamma})$ as modules for $\mathcal{S}_\beta \times \mathcal{S}_\eta$.*

Moreover, if $p^n > \lambda_1$ then $M^{\lambda+p^n\alpha}(P_\nu) \cong M^\lambda(P_\rho) \boxtimes M^{p^n\alpha}(P_{p^n\gamma})$ as modules for $\mathcal{S}_\beta \times \mathcal{S}_\eta$.

Proof. If $\mu = \sum_{i=0}^s \mu(i)p^i$ is the p -adic expansion of μ , and $\delta = \sum_{i=0}^l \delta(i)p^i$ is the p -adic expansion of δ , then since $n > s$, it follows that

$$(3) \quad \mu + p^n\delta = \sum_{i=0}^s \mu(i)p^i + \sum_{j=n}^{n+l} \delta(j-n)p^j$$

is the p -adic expansion of $\mu + p^n\delta$. The first part of (a) now follows easily from Grabmeier's description of a Young vertex, and the second part of (a) follows from Theorem 7(a).

By Proposition 11, $P_{p^n\gamma}$ is a vertex of $Y^{p^n\delta}$. Thus, applying Theorem 7 yields

$$(4) \quad Y^\mu(P_\rho) \cong Y^{\mu(0)} \boxtimes \dots \boxtimes Y^{\mu(s)} \text{ as a module for } \mathcal{S}_\beta$$

$$(5) \quad Y^{p^n\delta}(P_{p^n\gamma}) \cong Y^{\delta(0)} \boxtimes \dots \boxtimes Y^{\delta(l)} \text{ as a module for } \mathcal{S}_\eta.$$

We have already seen that P_ν is a vertex of $Y^{\mu+p^n\delta}$, and hence by Theorem 7 and the p -adic expansion given in (3), we see

$$Y^{\mu+p^n\delta}(P_\nu) \cong Y^{\mu(0)} \boxtimes \dots \boxtimes Y^{\mu(s)} \boxtimes Y^{\delta(0)} \boxtimes \dots \boxtimes Y^{\delta(l)}$$

as modules for $\mathcal{S}_{\beta \bullet \eta}$. Identifying $\mathcal{S}_{\beta \bullet \eta}$ with $\mathcal{S}_\beta \times \mathcal{S}_\eta$ and noting (4) and (5), we obtain the isomorphism in claim (b).

For each $i < p^n$, the number of orbits of \mathcal{S}_ν of length i is the same as the number of orbits of \mathcal{S}_ρ of length i (and these orbits are in one-to-one correspondence by shifting the entries by ap^n), and for each $i \geq p^n$, the orbits of \mathcal{S}_ν of length i are identified with the orbits of $\mathcal{S}_{p^n\gamma}$ of length i . Let \mathcal{B} be the basis for $M^{\lambda+p^n\alpha}(P_\nu)$ consisting of $\lambda+p^n\alpha$ -tabloids whose rows are unions of \mathcal{S}_ν -orbits. Let \mathcal{B}_λ and $\mathcal{B}_{p^n\alpha}$ be the corresponding tabloid bases for $M^\lambda(P_\rho)$, and $M^{p^n\alpha}(P_{p^n\gamma})$ respectively. If x_1, x_2 are tabloids lying in \mathcal{B}_λ and $\mathcal{B}_{p^n\alpha}$ respectively, then define a tabloid $x \in \mathcal{B}_{\lambda+p^n\alpha}$ by taking the i th row of x to be the union of the \mathcal{S}_ν -orbits corresponding to the $\mathcal{S}_{p^n\gamma}$ -orbits and \mathcal{S}_ρ -orbits occurring in the i th rows of x_1 and x_2 respectively. We have defined an injective map of $\mathcal{B}_\lambda \times \mathcal{B}_{p^n\alpha}$ into $\mathcal{B}_{\lambda+p^n\alpha}$. It is easily seen that if $p^n > \lambda_1$, then in fact this map is a bijection.

$\mathcal{S}_\beta \times \mathcal{S}_\eta$ acts by permuting the \mathcal{S}_ν -orbits of the same length amongst themselves, and \mathcal{S}_β and \mathcal{S}_η act on \mathcal{B}_λ and $\mathcal{B}_{p^n\alpha}$ by permuting the \mathcal{S}_ρ and $\mathcal{S}_{p^n\gamma}$ orbits of the same length amongst themselves. It follows that this embedding of $\mathcal{B}_\lambda \times \mathcal{B}_{p^n\alpha}$ into $\mathcal{B}_{\lambda+p^n\alpha}$ identifies a subset of $\mathcal{B}_{\lambda+p^n\alpha}$, on which $\mathcal{S}_\beta \times \mathcal{S}_\eta$ acts transitively. This subset therefore forms a basis for a direct summand N of $M^{\lambda+p^n\alpha}(P_\nu)$. Moreover, $N \cong M^\lambda(P_\rho) \boxtimes M^{p^n\alpha}(P_{p^n\gamma})$ as a module for $\mathcal{S}_\beta \times \mathcal{S}_\eta$. This completes the proof. \square

Theorem 13. *Let $a, r \in \mathbb{N}$ and let $\lambda, \mu \vdash r$, and $\alpha, \delta \vdash a$. Let μ have p -adic expansion $\mu = \sum_{i=0}^s \mu(i)p^i$. If $n > s$ then*

$$(6) \quad [M^{\lambda+p^n\alpha} : Y^{\mu+p^n\delta}] \geq [M^\lambda : Y^\mu][M^\alpha : Y^\delta].$$

Furthermore, if $p^n > \lambda_1$ then there is equality in (6).

Proof. Let ρ, γ be partitions such that \mathcal{S}_ρ and \mathcal{S}_γ are Young vertices of Y^μ and Y^δ respectively. By Proposition 12, a Young vertex of $Y^{\mu+p^n\delta}$ is \mathcal{S}_ν where $\nu = (p^n\gamma) \bullet \rho$. Let $P_\nu, P_{p^n\gamma}$, and P_ρ be Sylow p -subgroups of $\mathcal{S}_\nu, \mathcal{S}_{p^n\gamma}$, and \mathcal{S}_ρ respectively. By

Proposition 12, parts (b) and (c), we have

$$(7) \quad M^\lambda(P_\rho) \boxtimes M^{p^n\alpha}(P_{p^n\gamma}) \mid M^{\lambda+p^n\alpha}(P_\nu),$$

and

$$(8) \quad Y^\mu(P_\rho) \boxtimes Y^{p^n\delta}(P_{p^n\gamma}) \cong Y^{\mu+p^n\delta}(P_\nu)$$

as modules for $\mathcal{S}_\beta \times \mathcal{S}_\eta$. By Theorem 6, and the above discussion we have

$$\begin{aligned} [M^{\lambda+p^n\alpha} : Y^{\mu+p^n\delta}] &= [M^{\lambda+p^n\alpha}(P_\nu) : Y^{\mu+p^n\delta}(P_\nu)] \\ &\geq [M^\lambda(P_\rho) \boxtimes M^{p^n\alpha}(P_{p^n\gamma}) : Y^\mu(P_\rho) \boxtimes Y^{p^n\delta}(P_{p^n\gamma})] \\ &= [M^\lambda(P_\rho) : Y^\mu(P_\rho)][M^{p^n\alpha}(P_{p^n\gamma}) : Y^{p^n\delta}(P_{p^n\gamma})]. \end{aligned}$$

Thus, we conclude that

$$(9) \quad [M^{\lambda+p^n\alpha} : Y^{\mu+p^n\delta}] \geq [M^\lambda : Y^\mu][M^{p^n\alpha} : Y^{p^n\delta}].$$

Applying Theorem 1 gives $[M^{p^n\alpha} : Y^{p^n\delta}] = [M^\alpha : Y^\delta]$ and (6) follows. Of course, if $p^n > \lambda_1$, then by Proposition 12(c), and an identical argument to above, we obtain equality in (6). \square

We remark that in Theorem 13 we have given a sufficient condition for there to be equality in (6). However, this condition is not necessary, as demonstrated in the following theorem.

Theorem 14. *Let $a, r \in \mathbb{N}$ and let $\lambda, \mu \vdash r$. Let μ have p -adic expansion $\mu = \sum_{i=0}^s \mu(i)p^i$. If $p^n > \max(p^s, \lambda_2)$, where λ_2 is the second part of λ , then*

$$[M^{\lambda+(ap^n)} : Y^{\mu+(ap^n)}] = [M^\lambda : Y^\mu].$$

Proof. Let $a = \sum_{i \in \mathbb{N}} \lfloor a \rfloor_i p^i$ be the p -adic expansion of a . Let γ be the p -partition with $\lfloor a \rfloor_i$ parts equal to p^i for each i . Then \mathcal{S}_γ is a Young vertex of $Y^{(a)}$. We follow the notation of Proposition 12 with $\alpha = \delta = (a)$. So \mathcal{S}_ρ is a Young vertex of Y^μ , and \mathcal{S}_ν is the Young vertex of $Y^{\mu+(p^na)}$ where $\nu = (p^n\gamma) \bullet \rho$. Following the proof of Proposition 12, since $p^n > \lambda_2$, if x is a $\lambda + (p^na)$ -tabloid fixed under the action of \mathcal{S}_ν , then all \mathcal{S}_ν -orbits which coincide with $\mathcal{S}_{p^n\gamma}$ -orbits are of length at least p^n , and hence lie in the first row of x . It follows that $M^{\lambda+(p^na)}(P_\nu) \cong M^\lambda(P_\rho) \boxtimes M^{(p^na)}(P_{p^n\gamma})$ as modules for $\mathcal{S}_\beta \times \mathcal{S}_\eta$. Thus,

$$[M^{\lambda+(p^na)} : Y^{\mu+(p^na)}] = [M^{\lambda+(p^na)}(P_\nu) : Y^{\mu+(p^na)}(P_\nu)],$$

and hence noting that $Y^{\mu+(p^n a)}(P_\nu) \cong Y^\mu(P_\rho) \boxtimes Y^{(p^n a)}(P_{p^n \gamma})$, as modules for $\mathcal{S}_\beta \times \mathcal{S}_\eta$, it follows that

$$\begin{aligned} [M^{\lambda+(p^n a)} : Y^{\mu+(p^n a)}] &= [M^\lambda(P_\rho) \boxtimes M^{(p^n a)}(P_{p^n \gamma}) : Y^\mu(P_\rho) \boxtimes Y^{(p^n a)}(P_{p^n \gamma})] \\ &= [M^\lambda(P_\rho) : Y^\mu(P_\rho)][M^{(p^n a)}(P_{p^n \gamma}) : Y^{(p^n a)}(P_{p^n \gamma})] \\ &= [M^\lambda : Y^\mu][M^{(p^n a)} : Y^{(p^n a)}] \\ &= [M^\lambda : Y^\mu] \end{aligned}$$

where the last equality follows because $M^{(p^n a)} \cong Y^{(p^n a)}$ and the preceding equality is by the Broué correspondence. \square

Theorem 14 strengthens a known result:

Theorem 15 (Henke [9, Corollary 6.2]). *Let $r \in \mathbb{N}$, and $\lambda \vdash r$. If $\lambda_1 \geq r/2$ and $\lambda_2 < p^n$ then*

$$[M^{\lambda+(p^n a)} : Y^{\mu+(p^n a)}] = [M^\lambda : Y^\mu]$$

for every $\mu \vdash r$.

In many cases, dependent on the choice of λ and μ , Theorem 14 and Theorem 15 coincide. When $r, n \in \mathbb{N}$, and $\lambda \vdash r$ are such that $p^n > \lambda_2$ and $\lambda_1 \geq r/2$, then Theorem 14 is just a special case of Theorem 15. To see that Theorem 14 is not just a special case of Theorem 15, we consider a partition λ of r such that $\lambda_1 < r/2$. Then Theorem 15 does not apply, and Theorem 14 yields that for every n such that $p^n > \lambda_2$, we have

$$[M^{\lambda+(p^n a)} : Y^{\mu+(p^n a)}] = [M^\lambda : Y^\mu]$$

for every partition μ of r such that the highest exponent of p in the p -adic expansion of μ is less than n , and for every $a \in \mathbb{N}$.

As an example, we take the case $p = 3$ and $\lambda = (3, 2, 2)$. Then $r = 7$, and $\lambda_1 = 3$, so Theorem 15 does not apply. Applying Theorem 14, we obtain

$$(10) \quad [M^{(3+3^n a, 2, 2)} : Y^{\mu+(3^n a)}] = [M^{(3, 2, 2)} : Y^\mu]$$

whenever $\mu \vdash 7$ and every $a, n \in \mathbb{N}$ such that 3^n is larger than the highest exponent of 3 occurring in the 3-adic expansion of μ . In particular, if μ is 3-restricted, then the equality holds for all whenever $n \geq 1$.

3. THE INDECOMPOSABLE YOUNG PERMUTATION MODULES

In this section, we proceed to a proof of Theorem 2. The main method of proof is to demonstrate that a Young permutation module M^λ has composition factors lying in at least two different p -blocks of \mathcal{S}_r . For $p \geq 3$, in combination with a simple dimension argument, this is sufficient (Proposition 20), and similarly for the

case where $p = 2$ and r is odd (Lemma 21). For the situation where $p = 2$ and r is even, we apply a similar method, aside from a few exceptional cases which are dealt with separately, to reduce the problem of determining the indecomposable Young permutation modules to the case where $\lambda \vdash r$ has at most 2 parts. In this case, the p -Kostka numbers have been determined by Henke in [9], and an analysis of these multiplicities (Proposition 27) yields the final classification.

We recall the following well-known lemma:

Lemma 16. *Let G be a finite group and M a transitive permutation FG -module, where F is some field. Then M has a unique trivial submodule C . Moreover, C is a direct summand of M if, and only if, the characteristic of F does not divide $\dim_F M$.*

In particular, when F has characteristic zero, and M is the permutation module on the cosets of a Young subgroup \mathcal{S}_λ , Lemma 16 simply states that $\langle \xi^\lambda, \chi^{(r)} \rangle = 1$ for every $\lambda \vdash r$. This is a fact we will make repeated use of.

For notational convenience, from here on, all modules are defined over k , a field of characteristic p , and all results concerning ordinary representation theory (over fields of characteristic zero) are stated in terms of characters.

We note a useful first case that follows easily from Lemma 16.

Lemma 17. *$M^{(r-1,1)}$ is indecomposable if, and only if, p divides r .*

Proof. By Theorem 5 and Lemma 16, we have $M^{(r-1,1)} \cong Y^{(r-1,1)} \oplus \kappa Y^{(r)}$ where κ is either 0 or 1. By Lemma 16 the result follows, since $\dim_k M^{(r-1,1)} = r$. \square

We remind the reader that the ordinary character of a Young permutation module M^λ is denoted by ξ^λ and the ordinary irreducible characters are denoted by χ^λ for $\lambda \vdash r$. We denote by \langle , \rangle the usual inner product on characters.

We recall the induction product on modules (and characters) for symmetric groups. That is, if M is an \mathcal{S}_m -module, and N is an \mathcal{S}_n -module, then the induction product is defined by $M \bullet N = (M \otimes N)_{\mathcal{S}_m \times \mathcal{S}_n} \uparrow^{\mathcal{S}_{m+n}}$. Since this is well-known, the interested reader is referred to [15] for details. The induction product is both commutative and associative, and the decomposition of the induction product of two ordinary irreducible characters is given by the Littlewood–Richardson rule. For a description of the Littlewood–Richardson rule, and an algorithm for its application, see for example [13, 2.8.13 and 2.8.14].

We require the following facts:

Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition of r with $t \geq 2$. Then λ is the concatenation $\lambda = \gamma \bullet \delta$ of two partitions $\gamma = (\lambda_1, \dots, \lambda_i)$ and $\delta = (\lambda_{i+1}, \dots, \lambda_t)$. It follows that

$$(11) \quad \xi^\lambda = \xi^\gamma \bullet \xi^\delta,$$

and in particular, if χ_1 is a constituent of ξ^γ and χ_2 is a constituent of ξ^δ , then $\chi_1 \bullet \chi_2$ is a constituent of ξ^λ .

We recall also [11, Example 14.4] that if $d \leq \frac{r}{2}$, then

$$(12) \quad \xi^{(r-d,d)} = \sum_{i=0}^d \chi^{(r-i,i)}.$$

The key result is the following lemma:

Lemma 18. *Let $\lambda \vdash r$ then:*

- (a) *if $l(\lambda) \geq 2$ then $\langle \xi^\lambda, \chi^{(r-1,1)} \rangle > 0$;*
- (b) *if $r \geq 4$ and $l(\lambda) \geq 2$ with $\lambda \neq (r-1,1)$ then $\langle \xi^\lambda, \chi^{(r-2,2)} \rangle > 0$;*
- (c) *if $r \geq 5$ and $l(\lambda) \geq 3$ with $\lambda \neq (r-2,1,1)$ then $\langle \xi^\lambda, \chi^{(r-3,2,1)} \rangle > 0$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ with $t \geq 2$. Then $\xi^\lambda = \xi^{(\lambda_1, \dots, \lambda_{t-1})} \bullet \xi^{(\lambda_t)}$. From Lemma 16 follows that $\chi^{(r-\lambda_t)}$ is a constituent of $\xi^{(\lambda_1, \dots, \lambda_{t-1})}$, and hence $\chi^{(r-\lambda_t)} \bullet \chi^{(\lambda_t)} = \xi^{(r-\lambda_t, \lambda_t)}$ is a constituent of ξ^λ . Part (a) follows from (12).

For part (b): If $l(\lambda) = 2$ then the result follows from (12). For partitions λ of length $t \geq 3$, we note that by (a), $\chi^{(r-\lambda_t-1,1)}$ is a constituent of $\xi^{(\lambda_1, \dots, \lambda_{t-1})}$, and hence $\chi^{(r-\lambda_t-1,1)} \bullet \chi^{(\lambda_t)}$ is a constituent of $\xi^\lambda = \xi^{(\lambda_1, \dots, \lambda_{t-1})} \bullet \xi^{(\lambda_t)}$. By the Littlewood–Richardson rule, it can be checked that $\chi^{(r-2,2)}$ is a constituent of $\chi^{(r-\lambda_t-1,1)} \bullet \chi^{(\lambda_t)}$, and hence of ξ^λ , yielding part (b).

For part (c): From parts (a) and (b) and Lemma 16 follows that $\xi^{(s-2,2)}$ is a constituent of $\xi^{(\lambda_1, \dots, \lambda_{t-1})}$ where $s = \lambda_1 + \dots + \lambda_{t-1}$. Thus $\xi^{(s-2,2)} \bullet \xi^{(\lambda_t)}$ is a constituent of $\xi^{(\lambda_1, \dots, \lambda_{t-1})} \bullet \xi^{(\lambda_t)} = \xi^\lambda$. From the commutativity and associativity of the induction product follows

$$(13) \quad \xi^{(s-2,2)} \bullet \xi^{(\lambda_t)} = \xi^{(s-2, \lambda_t)} \bullet \xi^{(2)}.$$

Since $\xi^{(s-2, \lambda_t)}$ contains $\chi^{(r-3,1)}$, it follows that $\chi^{(r-3,1)} \bullet \chi^{(2)}$ is a constituent of ξ^λ . Recall that $\chi^{(r-3,1)} = \xi^{(r-3,1)} - \xi^{(r-2)}$, and hence

$$\chi^{(r-3,1)} \bullet \chi^{(2)} = (\xi^{(r-3,1)} - \xi^{(r-2)}) \bullet \xi^{(2)} = \xi^{(r-3,2,1)} - \xi^{(r-2,2)}.$$

By Young's rule $\xi^{(r-3,2,1)}$ contains $\chi^{(r-3,2,1)}$ but $\xi^{(r-2,2)}$ does not. Part (c) follows. \square

Recall that every positive integer r has a p -adic expansion, and we write this as $r = \sum_{i=0}^{\infty} \lfloor r \rfloor_i p^i$. So in particular $\lfloor r \rfloor_0 \in \{0, 1, \dots, p-1\}$ and $r \equiv \lfloor r \rfloor_0 \pmod{p}$. We note here that the p -core of (r) is $(\lfloor r \rfloor_0)$ and hence the principal p -block of \mathcal{S}_r is indexed by p -core $(\lfloor r \rfloor_0)$.

Lemma 19. *Let $r \in \mathbb{N}$ and let p be an odd prime. Then:*

- (a) $\chi^{(r-1,1)}$ lies in the principal p -block of \mathcal{S}_r if, and only if, p divides r ; and

(b) $\chi^{(r-2,2)}$ lies in the principal p -block of \mathcal{S}_r if, and only if, p divides $r - 1$.

Proof. This follows from calculating the relevant p -cores as follows: The p -core of $(r - 1, 1)$ is

$$\begin{cases} \emptyset & \text{if } p \text{ divides } r, \\ ([r - 1]_0, 1) & \text{if } [r]_0 \neq 0 \text{ and } [r - 1]_0 \neq 0, \\ (p, 1) & \text{if } [r]_0 \neq 0 \text{ and } [r - 1]_0 = 0. \end{cases}$$

The p -core of $(r - 2, 2)$ is

$$\begin{cases} (p + 1, 2) & \text{if } [r - 2]_0 = 1, \\ (1, 1) & \text{if } [r - 2]_0 = 0, \\ (1) & \text{if } [r - 2]_0 = p - 1, \\ ([r - 2]_0, 2) & \text{otherwise.} \end{cases}$$

□

We can now classify the indecomposable Young permutation modules over fields of odd characteristic.

Proposition 20. *Let p be an odd prime. Let $r \in \mathbb{N}$, and $\lambda \vdash r$. Then the following hold:*

- (a) *If p does not divide r then M^λ is indecomposable if, and only if, $\lambda = (r)$.*
- (b) *If p divides r , then M^λ is indecomposable if, and only if, $\lambda = (r - 1, 1)$, or $\lambda = (r)$.*

Moreover, when M^λ is decomposable, it has at least one indecomposable direct summand which does not lie in the principal block of \mathcal{S}_r .

Proof. Suppose p does not divide r . We may assume that $l(\lambda) \geq 2$. Then by Lemma 16 and Lemma 18, both $\chi^{(r-1,1)}$, and $\chi^{(r)}$ have non-zero multiplicity in ξ^λ . By Lemma 19 these two characters lie in different p -blocks. It follows that M^λ has composition factors lying in different blocks and hence is decomposable.

We now consider the case of p dividing r . By Lemma 17, the permutation module $M^{(r-1,1)}$ is indecomposable, and certainly $M^{(r)}$ is the trivial module and hence is indecomposable. For the case $r = 3$, and $p = 3$ this leaves only $M^{(1,1,1)} \cong k\mathcal{S}_3$ which is certainly decomposable (it decomposes into projective indecomposable summands $Y^{(2,1)}$ and $Y^{(1,1,1)}$ corresponding to the two non-isomorphic simple $k\mathcal{S}_3$ modules). The result therefore holds for $r = 3$. If $r \geq 4$ and $\lambda \neq (r), (r - 1, 1)$, then by Lemma 16 and Lemma 18 the characters $\chi^{(r-2,2)}$ and $\chi^{(r)}$ have non-zero multiplicity in ξ^λ . By Lemma 19, the characters $\chi^{(r-2,2)}$ and $\chi^{(r)}$ lie in different p -blocks. It follows that M^λ is decomposable. This completes the proof. □

Thus, for odd primes we have completed the classification. We are left to consider the less straightforward case of $p = 2$. We will reduce the problem to 2-part partitions by the same methods as in odd characteristic.

Lemma 21. *Let $p = 2$ and let $r \in \mathbb{N}$ be odd. Let $\lambda \vdash r$, then M^λ is indecomposable if, and only if, $\lambda = (r)$. Moreover, when $\lambda \neq (r)$, the permutation module has at least one indecomposable direct summand which does not lie in the principal block of \mathcal{S}_r .*

Proof. Suppose $\lambda \neq (r)$, then $l(\lambda) \geq 2$ so by Lemma 18 and Lemma 16 both $\chi^{(r-1,1)}$ and $\chi^{(r)}$ have non-zero multiplicity in ξ^λ . Since r is odd, the 2-core of $(r-1, 1)$ is $(2, 1)$, and hence $\chi^{(r-1,1)}$ and $\chi^{(r)}$ lie in different 2-blocks of \mathcal{S}_r . It follows that M^λ has composition factors from two different blocks, so M^λ is decomposable. \square

Proposition 22. *Let $p = 2$ and $r \in \mathbb{N}$ be even. Let $\lambda \vdash r$ such that $l(\lambda) \geq 3$. Then*

- (a) *M^λ is decomposable;*
- (b) *if $r \geq 6$ and $\lambda \neq (r-2, 1, 1)$ then M^λ has at least one indecomposable direct summand which does not lie in the principal 2-block of \mathcal{S}_r .*

Proof. First we address the case $r = 4$. This is a special case since \mathcal{S}_4 has just one 2-block. We have two partitions to consider, namely $(2, 1^2)$, and (1^4) . We note here again that $M^{(1^4)} \cong k\mathcal{S}_4$ and hence, since $k\mathcal{S}_4$ has 2 simple modules, this is decomposable with two non-isomorphic direct summands, the projective indecomposable $k\mathcal{S}_4$ -modules. The remaining partition $(2, 1^2)$ is a particular case of $(r-2, 1^2)$ for $r \geq 4$. We have shown in Example 9 that for r even and $r \geq 4$, we have

$$M^{(r-2,1,1)} \cong Y^{(r-2,1,1)} \oplus Y^{(r-1,1)}.$$

In particular $M^{(r-2,1,1)}$ is decomposable.

Assume now that $r \geq 6$ and r is even. Since $l(\lambda) \geq 3$, it follows from Lemma 18 and Lemma 16 that for $\lambda \neq (r-2, 1, 1)$ we have $\langle \xi^\lambda, \chi^{(r-3,2,1)} \rangle \neq 0$ and $\langle \xi^\lambda, \chi^{(r)} \rangle = 1$. The 2-core of $(r-3, 2, 1)$ is $(3, 2, 1)$, so $(r-3, 2, 1)$ does not lie in the principal block of $k\mathcal{S}_r$. Hence M^λ has direct summands occurring in different 2-blocks of \mathcal{S}_r . \square

Proposition 22 yields that in characteristic 2, if $r \in \mathbb{N}$ is even and $\lambda \vdash r$ with M^λ indecomposable then $\lambda = (r)$ or λ is a 2-part partition. Henke determined the p -Kostka numbers for 2-part partitions in [9].

Theorem 23 ([9, Theorem 3.3]). *Let $r \in \mathbb{N}$, and k be a field of characteristic p . If $0 \leq s \leq j \leq r/2$, then $Y^{(r-s,s)}$ is a direct summand of $M^{(r-j,j)}$ if, and only if, $\binom{r-2s}{j-s} \not\equiv 0 \pmod{p}$.*

Moreover, if $[M^{(r-j,j)} : Y^{(r-s,s)}] \neq 0$ then $[M^{(r-j,j)} : Y^{(r-s,s)}] = 1$.

In order to efficiently calculate binomial coefficients modulo a prime we use Lucas' Theorem which says the following:

$$\binom{r-2s}{j-s} \equiv \prod_i \binom{\lfloor r-2s \rfloor_i}{\lfloor j-s \rfloor_i} \pmod{p}.$$

For the remainder of this section, we fix $p = 2$.

For $m \in \mathbb{N}$ define $\nu(m)$ such that $2^{\nu(m)}$ is the highest power of 2 which divides m . So in particular $\nu(m) = \min\{i \in \mathbb{N} \mid \lfloor m \rfloor_i \neq 0\}$. Before proceeding to the final part of the classification, we give a preliminary example:

Example 24. Let $j \in \mathbb{N}$. If $0 \leq s < j$ then since $\lfloor 2(j-s) \rfloor_{\nu(j-s)} = 0$, it follows by Lucas' Theorem that $\binom{2(j-s)}{j-s}$ is even. Thus, $M^{(j,j)}$ is indecomposable.

Proposition 25. Let $j, r \in \mathbb{N}$ such that r is even and $r \geq 2j > 0$. Then $M^{(r-j,j)}$ is indecomposable if, and only if, 2^{n_j} divides $r - 2j$ where $n_j := \min\{\alpha \in \mathbb{N} \mid j < 2^\alpha\}$.

Furthermore, if 2^{n_j} does not divide $r - 2j$ then

$$[M^{(r-j,j)} : Y^{(r-j+2^{\nu(r-2j)}, j-2^{\nu(r-2j)})}] = 1.$$

Proof. We write $r = 2j + \beta$. We assume first that 2^{n_j} divides β . If $\beta = 0$ then by Example 24, the permutation module $M^{(j,j)}$ is indecomposable. Thus, we may assume that $\beta > 0$.

Since 2^{n_j} divides β , it follows that $\nu(\beta) \geq n_j$. Let $s \in \mathbb{N}$ be such that $0 \leq s \leq j-1$. Then we have

$$j-s \leq j < 2^{n_j} \leq 2^{\nu(\beta)} \leq \beta,$$

and hence $\nu(j-s) < \nu(\beta)$.

Now $r-2s = 2(j-s) + \beta$, and hence $\lfloor r-2s \rfloor_{\nu(j-s)} = \lfloor 2(j-s) \rfloor_{\nu(j-s)} = 0$. On the other hand, $\lfloor j-s \rfloor_{\nu(j-s)} = 1$, and hence

$$\binom{\lfloor r-2s \rfloor_{\nu(j-s)}}{\lfloor j-s \rfloor_{\nu(j-s)}} = 0.$$

By Lucas' Theorem, the binomial coefficient $\binom{r-2s}{j-s} \equiv 0 \pmod{2}$, and hence by Theorem 23, it follows that $[M^{(r-j,j)} : Y^{(r-s,s)}] = 0$. This holds for all $0 \leq s \leq j-1$. Therefore if $\nu(\beta) \geq n_j$ then $M^{(r-j,j)}$ is indecomposable.

Suppose now that 2^{n_j} does not divide β . Then $\beta > 0$ and $\nu(\beta) < n_j$. Define $s := j - 2^{\nu(\beta)}$, then $s \geq 0$. By Lucas' Theorem, the binomial coefficient

$$\binom{r-2s}{j-s} = \binom{\beta + 2^{\nu(\beta)+1}}{2^{\nu(\beta)}}$$

is odd, so by Theorem 23, the multiplicity $[M^{(r-j,j)} : Y^{(r-s,s)}]$ is equal to 1, and, in particular $M^{(r-j,j)}$ is decomposable. This completes the proof. \square

In the proof of Proposition 25 we have proved a statement concerning binomial coefficients and we state this as the following lemma:

Lemma 26. *Let $x, y \in \mathbb{N}$ with $x, y \geq 1$, and let $n_x := \min\{\alpha \in \mathbb{N} \mid x < 2^\alpha\}$. Then 2^{n_x} divides y if, and only if, $\binom{y+2^i}{i}$ is even for every $i = 1, \dots, x$. Moreover, if 2^{n_x} does not divide y , then $x \geq 2^{\nu(y)}$, and $\binom{y+2^{\nu(y)}+1}{2^{\nu(y)}}$ is odd.*

In Proposition 25, we have described all indecomposable Young permutation modules for the case $p = 2$ and r even, giving infinite families of two part partitions for which they correspond. We now give a (perhaps) more satisfying description, applying Proposition 25 to completely describe for a given even $r \in \mathbb{N}$, all the partitions λ of r such that M^λ is indecomposable:

Proposition 27. *Let $r, n \in \mathbb{N}$ such that r is even, and $2^n \leq r < 2^{n+1}$. Then there are precisely $n + 1$ partitions λ of r with the property that M^λ is indecomposable. These partitions are precisely the partition (r) and the partitions given for each $1 \leq i \leq n$ by $(r - k_i, k_i)$, where each k_i is the unique integer such that $2^{i-1} \leq k_i < 2^i$ and $k_i \equiv \frac{r-2^n}{2} \pmod{2^{i-1}}$.*

Proof. $M^{(r)} \cong k$ is an indecomposable Young permutation module. Define $\beta := r - 2^n$. If $0 < j \leq \frac{r}{2}$ then we recall from Proposition 25 that $M^{(r-j,j)}$ is indecomposable if, and only if, $r = 2j + 2^{n_j}\alpha$ where $2^{n_j-1} \leq j < 2^{n_j}$. Let $i, j \in \mathbb{N}$ such that $1 \leq i \leq n$ and $2^{i-1} \leq j < 2^i$. If $M^{(r-j,j)}$ is indecomposable then $2^n + \beta - j = r - j = j + 2^i\alpha$ for some $\alpha \in \mathbb{N}$. It follows that $j \equiv \frac{\beta}{2} \pmod{2^{i-1}}$. Also, if $j \equiv \frac{\beta}{2} \pmod{2^{i-1}}$ and $2^{i-1} \leq j < 2^i$, then j is unique and $r = 2j + 2^i\gamma$ for some $\gamma \in \mathbb{N}$. Hence $M^{(r-j,j)}$ is indecomposable. The result follows. \square

We note here that $k_1 = 1$, and hence we have again shown that $M^{(r-1,1)}$ is indecomposable whenever 2 divides r . Also, $k_n = r/2$ corresponding to Example 24.

There are some properties of the families of partitions labelling indecomposable permutation modules which are of interest. We outline some of these now.

By Theorem 1, it follows that if $\lambda, \mu \vdash r$, then $[M^\lambda : Y^\mu] = [M^{2\lambda} : Y^{2\mu}]$, and hence, if M^λ is indecomposable then $[M^{2\lambda} : Y^{2\mu}] = 0$ for every partition μ of r such that $\mu \neq \lambda$. To determine whether $M^{2\lambda}$ is indecomposable, would require to prove $[M^{2\lambda} : Y^\nu] = 0$ whenever $\nu \triangleright 2\lambda$ and at least one part of ν is odd. In the following proposition we apply Theorem 27 to the case when r is even, to show that $M^{2\lambda}$ must be indecomposable, and nearly all the indecomposable Young permutation modules for kS_{2r} occur in this way.

Proposition 28. *Let $r \in \mathbb{N}$ be even and let $\lambda \vdash r$. If M^λ is indecomposable then $M^{2\lambda}$ is indecomposable.*

Furthermore, there is a bijection between $\{\lambda \vdash r \mid M^\lambda \text{ is indecomposable}\}$ and $\{\mu \vdash 2r \mid M^\mu \text{ is indecomposable}\} \setminus \{(2r-1, 1)\}$ given by multiplication by 2.

Proof. Let n be such that $2^n \leq r < 2^{n+1}$ and define $\beta := r - 2^n$. By Proposition 27 there are precisely $n+1$ partitions λ of r such that M^λ is indecomposable. These partitions λ are given by (r) and $(r-k_i, k_i)$ where $2^{i-1} \leq k_i < 2^i$ and $k_i \equiv \frac{\beta}{2} \pmod{2^{i-1}}$ for $1 \leq i \leq n$. Then $2^i \leq 2k_i < 2^{i+1}$ and $2k_i \equiv \beta \pmod{2^i}$. Since multiplication by 2 produces the $n+1$ partitions $(2r)$ and $(2r-2k_i, 2k_i)$ for $1 \leq i \leq n$, an application of Proposition 27 to $2r$ gives the result. \square

Proposition 29. *Let $r \in \mathbb{N}$ be even and n be such that $2^n \leq r < 2^{n+1}$. Let $0 \leq j \leq r/2$ and let $n \leq k$, then $M^{(r-j,j)}$ is indecomposable if, and only if, $M^{(r+2^k-j,j)}$ is indecomposable.*

Proof. Write $r = 2^n + \beta$ with $0 \leq \beta < 2^n$. Then $m := r + 2^k = 2^k + 2^n + \beta$. An application of Proposition 27 then yields the result. \square

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